

The Space of Self-dual Connections on Quaternion Line Bundles of Instanton Number 1 over the Standard Euclidean 4-sphere

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May 2022

1 Introduction

In this note, we follow [Law85] to show that the space of t'Hooft connections gives a complete set of solutions to the self-dual equations on $E \rightarrow S^4$ of $SU(2)$ -bundles. We sometimes identify $SU(2)$ as the same as the space of quaternions \mathbb{H} and do not mention this explicitly. Firstly, we review some preliminaries about the definitions of connections and self-dual equations.

2 Preliminaries

Let E be any smooth real vector bundle over a differentiable manifold M , and denote by $\Gamma(E)$ the space of smooth cross sections of E . We denote the space of smooth p -forms with values in E by

$$\Omega^p(E) \equiv \Gamma(\wedge^p T^*M \otimes E). \quad (1)$$

Definition 2.1 (Connections) A *connection* on E is a linear map $\nabla : \Omega^0(E) \rightarrow \Omega^1(E)$ such that the Leibniz rule is satisfied, i.e

$$\nabla(f\phi) = df \otimes \phi + f\nabla\phi \quad (2)$$

for any $f \in C^\infty(M)$ and any $\phi \in \Omega^0(E)$.

Suppose E carries a metric, i.e., a inner product $\langle \cdot, \cdot \rangle$ smoothly defined on the fibres. This leads to the following definition

Definition 2.2 (Riemannian connections) A connection ∇ on E is said to be *riemannian* if for all sections $\phi_1, \phi_2 \in \Omega^0(E)$,

$$d\langle \phi_1, \phi_2 \rangle = \langle \nabla\phi_1, \phi_2 \rangle + \langle \phi_1, \nabla\phi_2 \rangle. \quad (3)$$

Since convex combination of riemannian connections is riemannian, a partition of unity argument shows that riemannian connections always exist.

If we have an $SU(2)$ -bundle E , then $SU(2)$ acts on smooth sections of E by multiplication, and we have the following definition:

Definition 2.3 ($SU(2)$ -connections) An $SU(2)$ -connection on E is a riemannian connection ∇ that is $SU(2)$ -linear, i.e. it commutes with scalar multiplication by elements in $SU(2)$ (quaternions).

Given two $SU(2)$ -connections ∇ and ∇' , we have $A \equiv \nabla - \nabla' \in \mathfrak{su}(2)$ in local coordinates, as can be seen by a calculation in local charts. We define two important bundles:

$$\mathfrak{G}_E \equiv \{L \in \text{Hom}_{SU(2)}(E, E) : L^T = -L\} \quad (4)$$

$$G_E \equiv \{L \in \text{Hom}_{SU(2)}(E, E) : L^T = L^{-1}\}. \quad (5)$$

Notice that elements of \mathfrak{G}_E and G_E can be represented by elements in $\mathfrak{su}(2)$ and $SU(2)$ locally. Moreover, there exists an exponential map $exp : \mathcal{Y} \rightarrow \mathcal{G}$, and \mathcal{Y} is a Lie algebra under pointwise bracket.

In general, we have a concrete characterization of the space of $SU(2)$ -connections given by the following proposition (see Proposition 5.1, [Law85]):

Proposition 2.1 *The space \mathcal{C} of $SU(2)$ -connections on E is an affine space (infinite dimensional) having $\Omega^1(\mathfrak{G}_E)$ as the vector group of translations.*

Finally, we have the definition for curvature:

Definition 2.4 (Curvature) *The **curvature** of a connection ∇ is the 2-form $R^\nabla \in \Omega^2(\text{Hom}(E, E))$ defined for smooth vector fields V, W by the rule*

$$R_{V,W} = \nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}. \quad (6)$$

3 Gauge Actions and the Self-Dual Yang-Mills Equations

For a metric quaternion line bundle E , We define the **gauge group** \mathcal{G} as the group of smooth bundle automorphisms preserving the metric and quaternion structure; i.e.

$$\mathcal{G} \equiv \Gamma(G_E) \quad (7)$$

There is an associated **gauge algebra** defined by

$$\mathcal{Y} \equiv \Gamma(\mathfrak{G}_E) \quad (8)$$

The gauge group \mathcal{G} acts naturally on the space \mathcal{C} of $SU(2)$ -connections by

$$\nabla^g \equiv g \circ \nabla \circ g^{-1} \quad (9)$$

A calculation shows

$$R^{\nabla^g} = g \circ R^\nabla \circ g^{-1} \quad (10)$$

Let E be a vector bundle with structure group G over a compact riemannian manifold M . We assume $G \subset O(m)$ and E carries an inner product compatible with G . To each connection $\nabla \in \mathcal{C}$ and its associated curvature form R^∇ , we can take its norm at a point x

$$\|R^\nabla\|_x^2 \equiv \sum_{i < j} \|R_{e_i, e_j}^\nabla\|_x^2, \quad (11)$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_x M$, and the norm of R_{e_i, e_j}^∇ is the usual one on $\text{Hom}(E, E)$, where $\langle A, B \rangle \equiv \text{trace}(A^T \circ B)$.

Definition 3.1 (Yang-Mills functional) *The Yang-Mills functional is the mapping $\mathcal{Y.M} : \mathcal{C} \rightarrow \mathbb{R}^+$*

$$\mathcal{Y.M}(\nabla) \equiv \frac{1}{2} \int_M \|R^\nabla\|^2, \quad (12)$$

Definition 3.2 (Yang-Mills connections) *A connection $\nabla \in \mathcal{C}$ is called a **Yang-Mills connection**, and its curvature R^∇ is called a **Yang-Mills field**, if $\text{grad}_\nabla(\mathcal{Y.M}) = 0$.*

Observe that the transformation formula for R^{∇^g} implies $\|R^{\nabla^g}\| = \|R^\nabla\|$, so Yang-Mills functional is gauge invariant and descends to a map $\mathcal{Y.M} : \mathcal{B} \rightarrow \mathbb{R}^+$, where \mathcal{B} is the quotient space \mathcal{C}/\mathcal{G} by gauge actions.

We now define the **Hodge star operator**. Let $(V, \langle \cdot, \cdot \rangle)$ be an oriented, 4-dimensional real inner product space and let (e_1, \dots, e_4) be an oriented orthonormal basis of V . Then there is a linear map $*$: $\wedge^2 V \rightarrow \wedge^2 V$,

defined by $*(e_i \wedge e_j) = e_k \wedge e_l$, where (i, j, k, l) is an even permutation of $(1, 2, 3, 4)$, and consequently $** = \text{Id}$, so there is an eigenspace decomposition of $\wedge^2 V = \Lambda_- \oplus \Lambda_+$ into spaces of vectors with eigenvalues 1 or -1 . We can define $*$ invariantly by the condition that

$$\phi \wedge * \psi = \langle \phi, \psi \rangle e_1 \wedge \dots \wedge e_4 \quad (13)$$

For M an oriented riemannian 4-manifold, we have a $*$ -operator $* : \wedge^2 T^* M \rightarrow \wedge^2 T^* M$ which extends to 2-forms with values in any bundle. Moreover, we have the splittings

$$\wedge^2 T^* M \otimes \mathfrak{G} = (\Lambda_+ \otimes \mathfrak{G}_E) \oplus (\Lambda_- \otimes \mathfrak{G}_E) \quad (14)$$

and

$$\Omega^2(\mathfrak{G}_E) = \Omega_+^2(\mathfrak{G}_E) \oplus \Omega_-^2(\mathfrak{G}_E) \quad (15)$$

In particular, we have the

Definition 3.3 (Self-dual connection) *A connection ∇ is called **self-dual** if $*R^\nabla = R^\nabla$*

We call solutions to the self-dual equations instantons. Instantons minimize the Yang-Mills functional in the sense of

$$\mathcal{Y.M}(\nabla) = \frac{1}{2} \int_M (\|R_+^\nabla\|^2 + \|R_-^\nabla\|^2) \quad (16)$$

$$4\pi^2 i(E) = \frac{1}{2} \int_M (\|R_+^\nabla\|^2 - \|R_-^\nabla\|^2), \quad (17)$$

where $2i(E) \equiv p_1(E)$ is the first Pontrjagin number of E (so for quaternion line bundles, $i(E)$ is the instanton number). Hence self-dual connections are automatically Yang-Mills connections.

Remark 3.1 (Conformal invariance of the self-dual connections) *The condition of self-duality is **conformally invariant** because the $*$ -operator is conformally invariant.*

4 Solutions on S^4

In this section, let $E \rightarrow S^4$ be the \mathbb{H} -linbe bundle of instanton number 1 over the standard euclidean 4-sphere. We follow the exposition in section III.4 [Law85] to give a complete set of explicit solutions to the self-dual equations on E .

We have the following result ([AHS78])

Proposition 4.1 *The points of the moduli space $\mathfrak{M} \cong SO_{5,1}/SO_4$ (the hyperbolic 5-space of constant negative sectional curvature in its unique invariant metric) represent in a one-to-one way all equivalence classes of self-dual connections on E .*

The moduli space \mathfrak{M} is 5-dimensional. We follow the exposition in section III.4 [Law85] to work out an explicit formula.

Fix the point $p = (0, \dots, 0, 1) \in S^4$, and let $S^4 - p \rightarrow \mathbb{R}^4$ be the stereographic projection. This gives conformal coordinates on S^4 , and the metric is of the form

$$ds^2 = 4|dx|^2/(1 + |x|^2)^2. \quad (18)$$

In particular, a connection is self-dual with respect to ds^2 iff it is self-dual with respect to the euclidean metric $|dx|^2$.

Consider a the trivialized bundle $\mathbb{R}^4 \times \mathbb{H} \rightarrow \mathbb{R}^4$ and write down a connection in the form

$$\nabla = d + A, \quad (19)$$

where A is a $\text{Im}(\mathbb{H})$ -valued 1-form. Identify \mathbb{R}^4 with \mathbb{H} and write our local coordinate as a quaternion $sx = x_0 + x_1l + x_2j + x_3k$. For each $\lambda \in \mathbb{R}^+$, we define

$$A^\lambda = \text{Im}\left(\frac{x \cdot d\bar{x}}{\lambda^2 + |x|^2}\right) \quad (20)$$

and compute the curvature form $R^\lambda = dA^\lambda + [A^\lambda, A^\lambda]$ given by

$$R^\lambda = \frac{\lambda^2 dx \wedge d\bar{x}}{(\lambda^2 + |x|^2)^2}, \quad (21)$$

One can check that this connection extends across the point pn to give a connection on the bundle E over S^4 . In fact, if we take stereographic projection from $-p$, we get coordinates y with $y = \frac{1}{x} = \frac{\bar{x}}{|x|^2}$. If over this coordinate chart we write down the connection $A^{\frac{1}{\lambda}}$, then $A^{\frac{1}{\lambda}}(y)$ and $A^\lambda(x)$ are related by the transition functions $u = \frac{x}{|x|} = \frac{\bar{y}}{|y|}$.

The calculation

$$\begin{aligned} -\frac{1}{2} dx \wedge d\bar{x} &= (dx_0 \wedge dx_1 + dx_2 \wedge dx_3)i + (dx_0 \wedge dx_2 + dx_4 \wedge dx_3)j \\ &\quad + (dx_0 \wedge dx_3 + dx_1 \wedge dx_2)k \end{aligned} \quad (22)$$

shows that R^λ is self-dual because its i, j, k components constitute an orthogonal basis of Λ_+^2 . In particular, when $\lambda = 1$, we see that R^1 represents the orthogonal projection $\Lambda^2 \rightarrow \Lambda_+^2$.

Also, observe that $A^\lambda = (r^\lambda)^* A^1$ and $R^\lambda = (r^\lambda)^* R^1$, where $r^\lambda : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is the dilation map

$$r^\lambda(x) = \frac{x}{\lambda}. \quad (23)$$

As $\lambda \rightarrow 0$, the connection and curvature become concentrated in smaller and smaller neighbourhoods of 0. In particular, with respect to the spherical metric, we have

$$\|R^\lambda\|^2 = \frac{3}{2} \lambda^4 \left(\frac{1 + |x|^2}{\lambda^2 + |x|^2}\right)^4. \quad (24)$$

Since the curvature form is gauge invariant, all connections ∇^λ are inequivalent for distinct λ .

Notice we can construct connections by choosing stereographic projection away from $-q$ for **any** point $q \in S^4$. For $\lambda < 1$, the density function $\|R^\lambda\|^2$ has a unique maximum at the point q , so each of these connections for distinct q and $0 < \lambda < 1$ are inequivalent. The association $(q, \lambda) \rightarrow \mathfrak{M}$ gives the standard diffeomorphism

$$S^4 \times (0, 1) \rightarrow B^5 - \{0\}, \quad (25)$$

where we have identified \mathfrak{M} with the Poincaré ball B^5 . Adding the symmetric connection A^1 (the same in all coordinates) completes the space.

References

- [AHS78] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. “Self-Duality in Four-Dimensional Riemannian Geometry”. In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 362.1711 (1978), pp. 425–461. ISSN: 00804630. URL: <http://www.jstor.org/stable/79638> (visited on 05/21/2022).
- [Law85] H. Blaine Jr. Lawson. “The theory of gauge fields in four dimensions”. In: 1985.